

COMBINATION OF FULLY QUASICONVEX SUBGROUPS AND ITS APPLICATIONS

WEN-YUAN YANG

ABSTRACT. In this paper, we state two combination theorems for relatively quasiconvex subgroups in a relatively hyperbolic group. Applications are given to the separability of double cosets of certain relatively quasiconvex subgroups and the existence of closed surface subgroups in relatively hyperbolic groups.

1. INTRODUCTION

The combination theorems for relatively hyperbolic groups have been developed by many authors, and possessed wide applications in the theory of relatively hyperbolic groups. See [3], [9], [1], [26] and [13], just to name a few. By contrast, the problems for combining relatively quasiconvex subgroups in a relatively hyperbolic group are less well-studied. This kind of combination results also afford many important applications in constructing good subgroups. For instance, the idea of combining compact surfaces with parallel boundaries was initiated in Freedman-Freedman [12] and further explored by Cooper, Long and Reid([7], [8]) to obtain closed surfaces in cusped hyperbolic 3-manifolds.

In a relatively hyperbolic group G , one can ask the following questions for the combination of relatively quasiconvex subgroups H, K :

- (1) Under what conditions the amalgamation $H \star_C K$ over $C = H \cap K$ is embedded in G as a relatively quasiconvex subgroup?
- (2) Under what conditions the HNN extension $H \star_{Q_1 \sim Q_2}$ over isomorphic subgroups Q_1, Q_2 is embedded in G as a relatively quasiconvex subgroup?

Recall that a relatively quasiconvex subgroup is itself a relatively hyperbolic group. Hence whether the combination of two relatively quasiconvex subgroups is relatively hyperbolic and also embedded into the ambient group G give two theoretical obstructions for solving the above problems.

In hyperbolic groups, Gitik showed that these two obstructions can be virtually eliminated in virtue of a separability property: two quasiconvex subgroups, if their intersection is separable, contain (many) finite index subgroups to generate a quasiconvex amalgamation [18]. Along this line, Martinez-Pedroza proved a combination theorem in a relatively hyperbolic group for combining relatively quasiconvex subgroups over a parabolic subgroup [23]. In the other direction, Baker-Cooper showed that a pair of geometrically finite subgroups with compatible parabolic subgroups can be virtually amalgamated [2].

Date: Dec 18, 2012.

2000 Mathematics Subject Classification. Primary 20F65, 20F67.

Key words and phrases. combination theorems, relatively quasiconvex groups, separability of double cosets, admissible paths.

This research is supported by the ERC starting grant GA 257110 RaWG.

In the present paper, we shall show two combination theorems in the same spirit for relatively quasiconvex subgroups with fully quasiconvex subgroups. Some applications of our combination results are given to the separability of double cosets and the existence of closed surface group. We now start by stating the combination theorems.

1. Combination theorems. Let G be a finitely generated group hyperbolic relative to a collection of subgroups \mathbb{P} . A *fully quasiconvex subgroup* H is relatively quasiconvex in G such that $P^g \cap H$ is either finite or of finite index in P^g for each $g \in G, P \in \mathbb{P}$. Fully quasiconvex subgroups generalize quasiconvex subgroups in hyperbolic groups and are receiving a great deal of attention in the study of relatively quasiconvex subgroups, see [22], [6] and [29] and [15].

Our first result is to deal with the (virtual) amalgamation of a relatively quasiconvex subgroup with a fully quasiconvex subgroup, generalizing results of Gitik in the hyperbolic case [18].

Theorem 1.1 (Virtual amalgamation). *Suppose H is relatively quasiconvex and K fully quasiconvex in a relatively hyperbolic group G . Then there exists a constant $D = D(H, K) > 0$ such that the following statements are true.*

- (1) *Let $\dot{H} \subset H$ and $\dot{K} \subset K$ be such that $\dot{H} \cap \dot{K} = C$ and $d(1, g) > D$ for any $g \in \dot{H} \cup \dot{K} \setminus C$, where $C = H \cap K$. Then $\langle \dot{H}, \dot{K} \rangle = \dot{H} \star_C \dot{K}$.*
- (2) *If \dot{H}, \dot{K} are, in addition, relatively quasiconvex, then $\langle \dot{H}, \dot{K} \rangle$ is relatively quasiconvex.*
- (3) *Moreover, every parabolic subgroup in $\langle \dot{H}, \dot{K} \rangle$ is conjugated into either \dot{H} or \dot{K} .*

Remark 1.2. Note that maximal parabolic subgroups are fully quasiconvex. In the case that K is a maximal parabolic subgroup, it suffices to assume $d(1, g) > D$ for any $g \in \dot{K} \setminus C$ in Theorem 1.1. This generalizes a result in [23].

We also obtain a combination theorem for glueing two parabolic subgroups of a relatively quasiconvex subgroup such that the HNN extension is relatively quasiconvex. Let $\Gamma^g = g\Gamma g^{-1}$ be a conjugate of a subgroup Γ in G .

Theorem 1.3 (HNN extension). *Suppose H is relatively quasiconvex in a relatively hyperbolic group G . Let $P \in \mathbb{P}$ and $f \in G$ such that $Q = P \cap H, Q' = Q^f$ are non-conjugate maximal parabolic subgroups in H . Then there exists a constant $D = D(H, P, f) > 0$ such that the following statements are true.*

- (1) *Suppose there exists $c \in P$ such that $Q^c = Q$ and $d(1, c) > D$ for any $g \in cQ$. Let $t = fc$. Then $\langle H, t \rangle = H \star_{Q^t=Q'} Q'$ is relatively quasiconvex.*
- (2) *Moreover, every parabolic subgroup in $\langle H, t \rangle$ is conjugated into H .*

Remark 1.4. The sufficiently long element c exists when Q is normal and of infinite index in P . In particular, this holds for the groups hyperbolic relative to abelian groups.

In the setting of hyperbolic 3-manifolds, Theorem 1.3 generalizes a theorem of Baker-Cooper [2, Theorem 8.8], which was used to glue parallel boundary components of immersed surfaces in a 3-manifold to construct closed surfaces in [8]. As an application of our theorem, we consider the existence of closed surface groups in a relatively hyperbolic group.

Let H be the fundamental group of a compact surface S . Recall that H has *no accidental parabolics* in a relatively hyperbolic group (G, \mathbb{P}) if the conjugacy class

of elements in H representing boundary components in S is exactly the elements in H which can be conjugated into some $P \in \mathbb{P}$.

Corollary 1.5. *Suppose that G is hyperbolic relative to abelian subgroups of rank at least two. Let H be the fundamental group of a compact surface with boundary such that H has no accidental parabolics in G . Then there exists a closed surface subgroup in G which is relatively quasiconvex.*

2. Our approach: admissible paths. The approach in proving our combination results is based on a notion of *admissible paths* in a geodesic metric space with a system of contracting subsets. A *contracting* subset is defined with respect to a preferred class of quasigeodesics such that any of them far from the contracting subset has a uniform bounded projection to it. See precise definitions in Section 2.

The notion of contracting subsets turns out to compass many interesting examples. For instance, quasigeodesics and quasiconvex subspaces in hyperbolic spaces, parabolic cosets in relatively hyperbolic groups [15], contracting segments in CAT(0) spaces [4] and the subgroup generated by a hyperbolic element in groups with nontrivial Floyd boundary [15]. Relevant to our context, it is worthwhile to point out that fully quasiconvex subgroups are contracting, as shown in [15, Proposition 8.2.4].

In terms of contracting subsets, an admissible path can be roughly thought as a concatenation of quasigeodesics which travels alternatively near contracting subsets and leave them in an orthogonal way (see Definition 2.12). The informal version of our main result about admissible paths is the following.

Proposition 1.6 (cf. Proposition 2.16). *Long admissible paths are quasigeodesics.*

Example 1.7. (1) Theorems 1.1 and 1.3 are proved by constructing admissible paths for each element in $H \star_C K$ and $H \star_{Q'} Q'$.

(2) Note that local quasigeodesics in hyperbolic spaces are admissible paths and hence quasigeodesics ([16], [19]).

(3) Since contracting segments in the sense of Bestvina-Fujiwara are contracting in our sense, Proposition 1.6 can be also thought as a unified version of Proposition 4.2 and Lemma 5.10 in [4].

3. Separability of double cosets. Recall that a subset X of a group G is *separable* if for any $g \in G \setminus X$, there exists a homomorphism ϕ of G to a finite group such that $\phi(g) \notin \phi(X)$; in other words, X is a *closed* subset in G with respect to the profinite topology. A group G is called *LERF* if every finitely generated subgroup are separable. A *slender* group contains only finitely generated subgroups.

An application of our combination theorem, generalizing Gitik [17] and Minasyan [25], is to give a criterion of separability of double cosets of certain relatively quasiconvex subgroups.

Theorem 1.8. *Suppose G is hyperbolic relative to slender LERF groups and separable on fully quasiconvex subgroups. Let H be relatively quasiconvex and K fully quasiconvex in G . If $H' \subset H, K' \subset K$ are relatively quasiconvex in G such that $H' \cap K'$ is of finite index in $H \cap K$, then $H'K'$ is separable.*

Remark 1.9. If H is also fully quasiconvex, then the condition on each parabolic subgroup being LERF and slender would not be necessary.

An interesting corollary is obtained as follows when each maximal parabolic subgroup are virtually abelian. Note that abelian groups are LERF and slender.

Corollary 1.10. *Suppose G is hyperbolic relative to virtually abelian groups and separable on fully quasiconvex subgroups. Then the double coset of any two parabolic subgroups is separable. In particular, the double coset of any two cyclic subgroups is separable.*

Remark 1.11. When G is the fundamental group of a cusped hyperbolic 3-manifold of finite volume, Hamilton-Wilton-Zalesskii showed in [20] that, without additional assumptions, the double coset of any two parabolic subgroups is separable. In [29], Wise proved that G is virtually special and thus separable on fully quasiconvex subgroups. Hence, this corollary with Wise's result gives another proof of their result.

This paper is structured as follows. In Section 2, we give a general development of the notion of admissible paths, which underlies the proofs of Theorems 1.1 and 1.3. Sections 3 & 4 and Section 5 are devoted to prove Theorems 1.1 and 1.3 respectively. In the final section, we give the proof of Theorem 1.8 and its corollary.

After the completion of this paper, the author noticed that Eduardo Martinez-Pedroza and Alessandro Sisto proved a more general combination theorem in [24]. Our Theorem 1.1 is a special case of their result. However, our methods are different and Theorem 1.3 does not follow from their result.

2. AXIOMATIZATION: ADMISSIBLE PATHS

The purpose of this section is two-fold. First, a notion of an admissible path is introduced as a model in proving our combination theorems in next sections. In fact, this notion arises an attempt to unify the proofs of combination theorems.

Secondly, we pay much attention to axiomatize the discussion, with the aim extracting the hyperbolic-like feature naturally occurred in various contexts. The motivating examples we have in mind are parabolic cosets in relatively hyperbolic groups and contracting segments in CAT(0) spaces.

2.1. Notations and Conventions. Let (Y, d) be a geodesic metric space. Given a subset X and a number $U \geq 0$, let $N_U(X) = \{y \in Y : d(y, X) \leq U\}$ be the closed neighborhood of X with radius U . Denote by $\|X\|$ the diameter of X with respect to d .

Fix a (sufficiently small) number $\delta > 0$ that won't change in the rest of paper. Given a point $y \in Y$ and subset $X \subset Y$, let $\Pi_X(y)$ be the set of points x in X such that $d(y, x) \leq d(y, X) + \delta$. Define the *projection* of a subset A to X as $\Pi_X(A) = \cup_{a \in A} \Pi_X(a)$.

Let p be a path in Y with initial and terminal endpoints p_- and p_+ respectively. Denote by $\ell(p)$ the length of p . Given two points $x, y \in p$, denote by $[x, y]_p$ the subpath of p going from x to y .

Let p, q be two paths in Y . Denote by $p \cdot q$ (or pq if it is clear in context) the concatenated path provided that $p_+ = q_-$.

A path p going from p_- to p_+ induces a first-last order as we describe now. Given a property (P), a point z on p is called the *first point* satisfying (P) if z is among the points w on p with the property (P) such that $\ell([p_-, w]_p)$ is minimal. The *last point* satisfying (P) is defined in a similar way.

Let $f(x, y) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$ be a function. For notational simplicity, we frequently write $f_{x,y} = f(x, y)$.

2.2. Contracting subsets.

Definition 2.1 (Contracting subset). Suppose \mathcal{L} is a preferred collection of quasi-geodesics in X . Let $\mu : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$ and $\epsilon : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$ be two functions.

Given a subset X in Y , if the following inequality holds

$$\|\Pi_X(q)\| < \epsilon(\lambda, c),$$

for any (λ, c) -quasigeodesic $q \in \mathcal{L}$ with $d(q, X) \geq \mu(\lambda, c)$, then X is called (μ, ϵ) -contracting with respect to \mathcal{L} . A collection of (μ, ϵ) -contracting subsets is referred to as a (μ, ϵ) -contracting system (with respect to \mathcal{L}).

Example 2.2. We note the following examples in various contexts.

- (1) Quasigeodesics and quasiconvex subsets are contracting with respect to the set of all quasigeodesics in hyperbolic spaces. These are best-known examples in the literature.
- (2) Fully quasiconvex subgroups (and in particular, maximal parabolic subgroups) are contracting with respect to the set of all quasigeodesics in the Cayley graph of relatively hyperbolic groups (see Proposition 8.2.4 in [15]). This is the main situation that we will deal with in Section 3.
- (3) The subgroup generated by a hyperbolic element is contracting with respect to the set of all quasigeodesics in groups with non-trivial Floyd boundary (see Proposition 8.2.4 in [15]). Here hyperbolic elements are defined in the sense of convergence actions on the Floyd boundary. Note that groups with non-trivial Floyd boundary include relatively hyperbolic groups [14], and it is not yet known whether these two classes of groups coincide.
- (4) Contracting segments in CAT(0)-spaces in the sense of in Bestvina-Fujiwara are contracting here with respect to the set of geodesics (see Corollary 3.4 in [4]).
- (5) Any finite neighborhood of a contracting subset is still contracting with respect to the same \mathcal{L} .

Convention. In view of examples above, the preferred collection \mathcal{L} in the sequel is always assumed to be containing all geodesics in Y .

Definition 2.3 (Quasiconvexity). Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}_+$ be a function. A subset $X \subset Y$ is called σ -quasiconvex if given $U \geq 0$, any geodesic with endpoints in $N_U(X)$ lies in the neighborhood $N_{\sigma(U)}(X)$.

Quasiconvexity follows from the above contracting property.

Lemma 2.4. Let X be a (μ, ϵ) -contracting subset in Y . Then there exists a function $\sigma : \mathbb{R} \rightarrow \mathbb{R}_+$ such that X is σ -quasiconvex.

Proof. Given $U \geq 0$, let γ be a geodesic with endpoints in $N_U(X)$. Define $\sigma(U) = 3 \max(U, \mu_{1,0}) + \epsilon_{1,0}$. It suffices to verify that $\gamma \subset N_{\sigma(U)}(X)$.

Let z be a point in γ such that $d(z, X) \geq \mu_{1,0}$. Denote by p the maximal connected segment of γ containing z such that $d(p, X) \geq \mu(1, 0)$. Then $\|\Pi_X(p)\| < \epsilon_{1,0}$. Note that $d(p_-, X) \leq \max(U, \mu_{1,0})$. Hence, it follows that

$$d(z, X) \leq d(z, p_-) + d(p_-, X) \leq \sigma(U),$$

which finishes the proof. \square

We need a notion of orthogonality of a quasigeodesic path to a contracting subset.

Definition 2.5 (Orthogonality). Let X be a (μ, ϵ) -contracting subset in Y . Given a function $\tau : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$, a (λ, c) -quasigeodesic p is said to be τ -orthogonal to X if $\|p \cap N_{\mu(\lambda, c)}(X)\| \leq \tau(\lambda, c)$.

The main point of an orthogonal path is that its projection to the contracting subset is uniformly bounded. In particular, the following fact will be frequently used later without explicit mention.

Lemma 2.6. *Given a (ϵ, μ) -contracting subset X , let q be a (λ, c) -quasigeodesic in \mathcal{L} that is τ -orthogonal to X . Then the following inequality holds*

$$\|\Pi_X(q)\| < A_{\lambda, c},$$

where

$$(1) \quad A_{\lambda, c} = \mu(\lambda, c) + \tau(\lambda, c) + \epsilon(\lambda, c).$$

We now shall introduce an additional property, named bounded intersection property for a contracting system.

Definition 2.7 (Bounded Intersection). Given a function $\nu : \mathbb{R} \rightarrow \mathbb{R}_+$, two subsets $X, X' \subset Y$ have ν -bounded intersection if the following inequality holds

$$\|N_U(X) \cap N_U(X')\| < \nu(U)$$

for any $U \geq 0$.

Remark 2.8. Typical examples include sufficiently separated quasiconvex subsets in hyperbolic spaces, and parabolic cosets in relatively hyperbolic groups (see Lemma 3.1).

A (μ, ϵ) -contracting system \mathbb{X} is said to have ν -bounded intersection if any two distinct $X, X' \in \mathbb{X}$ have ν -bounded intersection. A related notion is the following bounded projection property, which is equivalent to the bounded intersection property under the contracting assumption as in Lemma 2.10 below.

Definition 2.9 (Bounded Projection). Two subsets $X, X' \subset Y$ have B -bounded projection for some $B > 0$ if the following holds

$$\|\Pi_X(X')\| < B, \|\Pi_{X'}(X)\| < B$$

Lemma 2.10 (Bounded intersection \Leftrightarrow Bounded projection). *Let X, X' be two (μ, ϵ) -contracting subsets. Then X, X' have ν -bounded intersection for some $\nu : \mathbb{R} \rightarrow \mathbb{R}_+$ if and only if they have B -bounded projection for some $B > 0$.*

Proof. \Rightarrow : Let $z, w \in \Pi_X(X')$ be such that $d(z, w) = \|\Pi_X(X')\|$. Then there exist $\hat{z}, \hat{w} \in X'$ which project to z, w respectively. Set $B = 2\epsilon_{1,0} + \nu(\mu_{1,0})$.

Let's consider $d(z, X) > \mu_{1,0}$ and $d(w, X) > \mu_{1,0}$. Other cases are easier. Let p be a geodesic segment between z, w . Let \hat{u}, \hat{v} be the first and last points respectively on p such that $d(\hat{u}, X) \leq \mu_{1,0}, d(\hat{v}, X) \leq \mu_{1,0}$. Let u, v be a projection point of \hat{u}, \hat{v} to X respectively. Then $d(u, v) \leq \nu(\mu_{1,0})$ by the ν -bounded intersection of X, X' . Since X is a (μ, ϵ) -contracting subset, we obtain that $d(z, u) \leq \epsilon_{1,0}, d(w, v) \leq \epsilon_{1,0}$. Hence

$$d(z, w) \leq d(z, u) + d(u, v) + d(v, w) \leq B.$$

\Leftarrow : Given $U > 0$, let $z, w \in N_U(X) \cap N_U(X')$. Let $\hat{z}, \hat{w} \in X'$ be such that $d(\hat{z}, z) \leq U, d(\hat{w}, w) \leq U$. Project z, w to $z', w' \in X$ respectively. Then $d(z, w) \leq d(z', w') + 2U$. It remains to bound $d(z', w')$.

It is easy to verify that the projection of a geodesic segment of length U on X have a upper bounded size $(2\mu_{1,0} + \epsilon_{1,0} + U)$. Hence $\|\Pi_X([\hat{z}, z])\| \leq (2\mu_{1,0} + \epsilon_{1,0} + U)$, $\|\Pi_X([\hat{w}, w])\| \leq (2\mu_{1,0} + \epsilon_{1,0} + U)$. It follows that

$$\begin{aligned} d(z', w') &\leq \|\Pi_X([\hat{z}, z])\| + \|\Pi_X(X')\| + \|\Pi_X([\hat{w}, w])\| \\ &\leq B + 2(2\mu_{1,0} + \epsilon_{1,0} + U). \end{aligned}$$

Then $d(z, w) \leq B + 4\mu_{1,0} + 2\epsilon_{1,0} + 2U$. It suffices to set $\nu(U) = B + 4\mu_{1,0} + 2\epsilon_{1,0} + 2U$. \square

To conclude this subsection, we note a thin-triangle property when one side of a triangle lies near a contracting subset. Recall that the constant $A_{\lambda,c}$ below is defined in (1).

Lemma 2.11. *Given $X\lambda \geq 1, c \geq 0$, let $\gamma = pq$, where p is a geodesic and q is (λ, c) -quasigeodesic in \mathcal{L} . Assume that $p_-, p_+ \in X \in \mathbb{X}$ and q is τ -orthogonal to X . Then γ is a $(\lambda, C_{\lambda,c})$ -quasigeodesic, where*

$$(2) \quad C_{\lambda,c} = \lambda(\mu_{\lambda,c} + \epsilon_{\lambda,c} + A_{\lambda,c}) + c.$$

Proof. Let α be a geodesic such that $\alpha_- = \gamma_-$, $\alpha_+ = \gamma_+$. Let z be the last point on α such that $d(z, X) \leq \mu_{\lambda,c}$. Project z to a point z' on X . Then $d(z, z') \leq \mu_{\lambda,c}$. By projection, we have

$$\begin{aligned} d(z, q_-) &\leq \|\Pi_X([\alpha_-, z]_\alpha)\| + \|\Pi_X(q)\| \\ &\leq \epsilon_{1,0} + A_{\lambda,c}. \end{aligned}$$

Then we have

$$\begin{aligned} d(z, q_-) &\leq d(q_-, z) + d(z, z') \\ &\leq \mu_{1,0} + \epsilon_{1,0} + A_{\lambda,c}. \end{aligned}$$

Hence $\ell(\gamma) = \ell(p) + \ell(q) < \lambda d(\gamma_-, \gamma_+) + c \leq \lambda d(\gamma_-, \gamma_+) + C_{\lambda,c}$. \square

2.3. Admissible Paths. In this subsection, we give the precise definition of an admissible path, which is roughly a piecewise quasigeodesic path with well-controlled local properties.

Recall that \mathcal{L} is a preferred collection of quasigeodesics in X such that \mathcal{L} contains all geodesics. In what follows, let \mathbb{X} be a (μ, ϵ) -contracting system in Y with respect to \mathcal{L} . Then each $X \in \mathbb{X}$ is σ -quasiconvex, where σ is given by Lemma 2.4.

Fix also two functions $\nu : \mathbb{R} \rightarrow \mathbb{R}_+$ and $\tau : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$, which the reader may have in mind are the bounded intersection function and orthogonality function respectively.

Definition 2.12 (Admissible Paths). Given $D \geq 0, \lambda \geq 1, c \geq 0$, a (D, λ, c) -admissible path γ is a concatenation of (λ, c) -quasigeodesics in Y such that the following conditions hold:

- (1) Exactly one quasigeodesic p_i of any two consecutive ones in γ has two endpoints in a contracting subset $X_i \in \mathbb{X}$,
- (2) Each p_i has length bigger than $\lambda D + c$, except that p_i is the first or last quasigeodesic in γ ,
- (3) For each X_i , the quasigeodesics with one endpoint in X_i are τ -orthogonal to X_i , and
- (4) Either any two X_i, X_{i+1} (if defined) have ν -bounded intersection, or the quasigeodesic q_{i+1} between them has length bigger than $\lambda D + c$.

Remark 2.13. Note that if \mathbb{X} has ν -bounded intersection, then the condition (3) is always satisfied.

For definiteness in the sequel, usually write $\gamma = p_0 q_1 p_1 \dots q_n p_n$ and assume that p_i has endpoints in a contracting subset $X_i \in \mathbb{X}$ and the following conditions hold.

- (1) $\ell(p_i) > \lambda D + c$ for $0 < i < n$.
- (2) $q_i \in \mathcal{L}$ is τ -orthogonal to both X_{i-1} and X_i for $1 \leq i \leq n$.
- (3) For $1 \leq i \leq n$, either $\ell(q_i) > \lambda D + c$, or X_{i-1} and X_i have ν -bounded intersection.

Remark 2.14. The collection $\{X_i\}$ therein will be referred to as the (associated) contracting subsets for γ . It is not required that $X_i \neq X_j$ for $i \neq j$. This often facilitates the verification of a path being admissible.

Definition 2.15 (Fellow Traveller). Assume that $\gamma = p_0 q_1 p_1 \dots q_n p_n$ is a (D, λ, c) -admissible path, where each p_i has two endpoints in $X_i \in \mathbb{X}$. Let α be a path such that $\alpha_- = \gamma_-$, $\alpha_+ = \gamma_+$.

Given $R > 0$, the path α is a *R-fellow traveller* for γ if there exists a sequence of successive points z_i, w_i ($0 \leq i \leq n$) on α such that $d(z_i, w_i) \geq 1$ and $d(z_i, (p_i)_-) < R$, $d(w_i, (p_i)_+) < R$.

2.4. Quasi-geodesicity of long admissible paths. The aim of this subsection is to show that for a sufficiently large D , a (D, λ, c) -admissible path is a quasi-geodesic. The main technical result of this subsection can be stated as follows.

Proposition 2.16. *Given $\lambda \geq 1, c \geq 0$, there are constants $D = D(\lambda, c) > 0, R = R(\lambda, c) > 0$ such that the following statement holds.*

*Let γ be a (D_0, λ, c) -admissible path for $D_0 > D$. Then any geodesic α between γ_- and γ_+ is a *R-fellow traveller* for γ .*

The main corollary is that a long admissible path is a quasi-geodesic.

Corollary 2.17. *Given $\lambda \geq 1, c \geq 0$, there are constants $D = D(\lambda, c) > 0, \Lambda = \Lambda(\lambda, c) \geq 1$ such that given any $D_0 > D$ the (D_0, λ, c) -admissible path is a $(\Lambda, 0)$ -quasi-geodesic.*

Proof. Let $D = D(\lambda, c)$, $R = R(\lambda, c)$ be given by Proposition 2.16. Then it suffices to set $\Lambda = \lambda(6R + 1) + 3c$ to complete the proof. \square

The remainder of this subsection is devoted to the proof of Proposition 2.16.

We now define, a priori, the candidate constants which are calculated in the course of proof:

$$R = R(\lambda, c) = \max\{(6), (8), (14)\},$$

and

$$D = D(\lambda, c) = \max\{(4), (5), (7), (9), (13)\}$$

Let $\gamma = p_0 q_1 p_1 \dots q_n p_n$ be a (D_0, λ, c) -admissible path for $D_0 > D$, where p_i, q_i are (λ, c) -quasi-geodesics. For definiteness, assume that each p_i has endpoints in $X_i \in \mathbb{X}$. Moreover, we can assume that each p_i is a geodesic, as the general case follows as a direct consequence.

The proof of Proposition 2.16 is achieved by the induction on the number of contracting subsets $\{X_i\}$ for γ .

We start with a lemma describing the subpath of an admissible path around a contracting subset. Denote by $\Pi_k(q)$ the projection of q to X_k .

Lemma 2.18 (Near contracting subsets). *Let X_k ($0 \leq k \leq n$) be a contracting subset for γ . Then we have the following*

$$\forall k > 0 : \|\Pi_k(p_{k-1}q_k)\| < B_{\lambda,c},$$

and

$$\forall k < n : \|\Pi_k(q_{k+1}p_{k+1})\| < B_{\lambda,c},$$

where

$$B_{\lambda,c} = 2\epsilon_{1,0} + 2\mu_{1,0} + \nu(\mu_{1,0} + \sigma_0) + A_{\lambda,c}.$$

Proof. We only prove the inequality for the case $p_{k-1}q_k$. The other case is similar. We claim the following inequality

$$(3) \quad \|p_{k-1} \cap N_{\mu(1,0)}(X_k)\| < \nu(\mu_{1,0} + \sigma_0),$$

from which the conclusion follows. In fact, assuming the inequality (3) is true. Let z (resp. w) be the first (resp. last) point of p_{k-1} such that $z, w \in N_{\mu(1,0)}(X_k)$. Then we have

$$\begin{aligned} \|\Pi_k(p_{k-1}q_k)\| &< \|\Pi_k([(p_{k-1})_-, z]_{p_{k-1}})\| + \|\Pi_k([z, w]_{p_{k-1}})\| \\ &\quad + \|\Pi_k([w, (p_{k-1})_+]_{p_{k-1}})\| + \|\Pi_k(q_k)\| \\ &< 2\epsilon_{1,0} + (2\mu_{1,0} + \nu(\mu_{1,0} + \sigma_0)) + A_{\lambda,c} < B_{\lambda,c}. \end{aligned}$$

In order to prove (3), we examine the following two cases by the definition of an admissible path.

Case 1: $\ell(q_k) > \lambda D + c$. We show that $p_{k-1} \cap N_{\mu(1,0)}(X_k) = \emptyset$ and hence (3) holds trivially. Suppose not. Let w be the last point on p_{k-1} such that $d(w, X_k) \leq \mu_{1,0}$. Project w to a point $w' \in X_k$. Then $d(w, w') < \mu_{1,0}$. Using projection, we obtain

$$\begin{aligned} d(w, (q_k)_+) &< d(w, w') + d(w', (q_k)_+) \\ &< \mu_{1,0} + \|\Pi_k[w, p_{k-1}]_{p_{k-1}}\| + \|\Pi_k(q_k)\| \\ &< \mu_{1,0} + \epsilon_{1,0} + A_{\lambda,c}. \end{aligned}$$

Since $p_{k-1}q_k$ is a $(\lambda, C_{\lambda,c})$ -quasigeodesic by Lemma 2.11, we have that

$$C_{\lambda,c} + \lambda d(w, (q_k)_+) > \ell([w, (p_{k-1})_+]_{p_{k-1}}) + \ell(q_k).$$

As it is assumed that

$$(4) \quad D > \mu_{1,0} + \epsilon_{1,0} + A_{\lambda,c} + C_{\lambda,c},$$

this gives a contradiction with $\ell(q_k) > \lambda D + c$.

Case 2: Otherwise X_{k-1}, X_k have ν -bounded intersection. Then p_{k-1} lies in $N_{\sigma(0)}(X_{k-1})$. By the bounded projection property, we have

$$\|p_{k-1} \cap N_{\mu(1,0)}(X_k)\| < \|N_{\sigma(0)}(X_{k-1}) \cap N_{\mu(1,0)}(X_k)\| < \nu(\mu_{1,0} + \sigma_0).$$

This establishes (3). \square

We are ready to start the base step of induction.

Lemma 2.19 (Base Step). *Proposition 2.16 is true for $n = 1$ and $n = 2$.*

Proof. We shall prove a slightly stronger result: let α be a geodesic such that $d(\alpha_-, \gamma_-) \leq \mu_{1,0}$, $d(\alpha_+, \gamma_+) \leq \mu_{1,0}$, then α is a R -fellow traveller for γ .

The case " $n = 1$ ". Assume that $\gamma = q_1p_1q_2$, where the geodesic p_1 has two endpoints in a contracting subset X_1 .

Note that $[\alpha_-, \gamma_-]$ and $[\alpha_+, \gamma_+]$ are of length at most $\mu_{1,0}$. By projection we have

$$\|\Pi_1([\alpha_-, \gamma_-])\| \leq \epsilon_{1,0} + 3\mu_{1,0}, \quad \|\Pi_1([\alpha_+, \gamma_+])\| \leq \epsilon_{1,0} + 3\mu_{1,0}.$$

We claim that $N_{\mu(1,0)}(X_1) \cap \alpha \neq \emptyset$. Suppose not. Then we can estimate by projection

$$\begin{aligned} \ell(p_1) &\leq \|\Pi_1(q_1)\| + \|\Pi_1(\alpha)\| + \|\Pi_1(q_2)\| + \|\Pi_1([\alpha_-, \gamma_-])\| + \|\Pi_1([\alpha_+, \gamma_+])\| \\ &\leq 2A_{\lambda,c} + 3\epsilon_{1,0} + 6\mu_{1,0}. \end{aligned}$$

This gives a contradiction as it is assumed that

$$(5) \quad D > 2A_{\lambda,c} + 3\epsilon_{1,0} + 6\mu_{1,0}.$$

Let z and w be the first and last points of α such that $z, w \in N_{\mu(1,0)}(X_1)$. Project z, w to z', w' to X_1 respectively. Hence we see

$$\begin{aligned} d((q_1)_+, z) &\leq \|\Pi_1(q_1)\| + \|\Pi_1([\alpha_-, z]_\alpha)\| + \|\Pi_1([\alpha_-, \gamma_-])\| + d(z, z') \\ &\leq A_{\lambda,c} + 2\epsilon_{1,0} + 4\mu_{1,0} < R - 1, \end{aligned}$$

as it is assumed that

$$(6) \quad R > A_{\lambda,c} + 2\epsilon_{1,0} + 4\mu_{1,0} + 1.$$

It is similar that $d((q_2)_-, w) < R - 1$. Up to a slight modification of z, w , we see that α is a R -fellow traveller for γ .

The case " $n = 2$ ". This case is similar to the case " $n=1$ ". We only indicate the necessary changes in the below.

Assume that $\gamma = q_1 p_1 q_2 p_2 q_3$, where the geodesic p_1, p_2 have two endpoints in contracting subsets X_1, X_2 respectively.

We first claim that $N_{\mu(\lambda,c)}(X_1) \cap q_3 = \emptyset$. If not, let z be the first point on q_3 such that $z \in N_{\mu(\lambda,c)}(X_1) \cap q_3$. Project z to z' on X_1 . By Lemma 2.18, we see that

$$d(z', (q_2)_-) \leq \|\Pi_1(q_2 p_2)\| + \|\Pi_1([(q_3)_-, z]_{q_3})\| \leq B_{\lambda,c} + \epsilon_{\lambda,c}.$$

The case " $n = 1$ " shows that $q_2 p_2 q_3$ is a $(\Lambda, 0)$ -quasigeodesic, where $\Lambda = \Lambda(\lambda, c)$ is given by Corollary 2.17. It follows that

$$\ell(p_2) \leq \ell([(q_2)_-, z]_\gamma) \leq \Lambda d((q_2)_-, z) \leq \Lambda(B_{\lambda,c} + \epsilon_{\lambda,c} + \mu_{\lambda,c}).$$

This gives a contradiction as it is assumed that

$$(7) \quad D > \Lambda(B_{\lambda,c} + \epsilon_{\lambda,c} + \mu_{\lambda,c}).$$

Hence it is shown that $N_{\mu(\lambda,c)}(X_1) \cap q_3 = \emptyset$.

Using the same argument as the case " $n = 1$ ", we can see that $\alpha \cap N_{\mu(1,0)}(X_i) \neq \emptyset$ for $i = 1, 2$. Let z_1 and w_1 be the first and last points of α such that $z_1, w_1 \in N_{\mu(1,0)}(X_1)$. Then as in the case " $n = 1$ ", we obtain

$$d((p_1)_-, z_1) \leq A_{\lambda,c} + 2\epsilon_{1,0} + 4\mu_{1,0} < R - 1,$$

and

$$\begin{aligned} d((p_1)_+, w_1) &\leq \|\Pi_1(q_2 p_2)\| + \|\Pi_1(q_3)\| + \|\Pi_1([w_1, \alpha_+]_\alpha)\| + \|\Pi_1([\alpha_+, \gamma_+])\| \\ &\leq B_{\lambda,c} + 3\epsilon_{1,0} + 4\mu_{1,0} < R - 1, \end{aligned}$$

as it is assumed that

$$(8) \quad R > B_{\lambda,c} + 3\epsilon_{1,0} + 4\mu_{1,0} + 1.$$

Consider the path $\gamma' = [w'_1, (p_1)_+] q_2 p_2 q_3$ which is (D, λ, c) -admissible. Let $\alpha' = [w_1, \alpha_+]_\alpha$. Let z_2 and w_2 be the first and last points of α' such that $z_2, w_2 \in N_{\mu(1,0)}(X_2)$. Similarly as above, we obtain that $d((p_2)_-, z_2) \leq R - 1, d((p_2)_+, w_2) \leq R - 1$.

Consequently, it is shown that α is a R -fellow traveller for γ . \square

Inductive Assumption: Assume that Proposition 2.16 holds for any (D_0, λ, c) -admissible path γ' with k contracting subsets $X_i \in \mathbb{X}$ ($k \leq n$). Then any geodesic between γ'_-, γ'_+ is a R -fellow traveller for γ' . Moreover γ' is a $(\Lambda, 0)$ -quasigeodesic, where $\Lambda = \Lambda(\lambda, c)$ is given by Corollary 2.17.

We now consider the admissible path $\gamma = p_0 q_1 p_1 \dots q_n p_n$, which has $n + 1$ contracting subsets $\{X_i \in \mathbb{X} : 0 \leq i \leq n\}$.

Lemma 2.20 (Far from contracting subsets). *Let X_k ($0 < k < n$) be a contracting subset for γ . Denote $\beta = [(p_0)_-, (q_{k-2})_+]_\gamma$. Then the following holds*

$$\beta \cap N_{R+\sigma(\mu(1,0))}(X_k) = \emptyset.$$

Proof. Suppose, to the contrary, that $\beta \cap N_{R+\sigma(\mu(1,0))}(X_k) \neq \emptyset$. Let z be the last point on β such that $d(z, X_k) \leq R + \sigma_{\mu(1,0)}$. Project z to w on X_k .

Observe that $\hat{\beta} = \beta \cup p_{k-1} q_k [(p_k)_-, w]$ is a (D_0, λ, c) -admissible path with at most n contracting subsets, as $k < n$. Hence $\hat{\beta}$ is a $(\Lambda, 0)$ -quasigeodesic by Inductive Assumption. It follows that

$$\ell([z, w]_{\hat{\beta}}) < \Lambda(R + \sigma_{\mu(1,0)}).$$

This gives a contradiction with $\ell(p_{k-1}) > D$, as it is assumed that

$$(9) \quad D > \Lambda(R + \sigma_{\mu(1,0)}).$$

Therefore, the segment β has at least a distance $R + \sigma_{\mu(1,0)}$ to X_k . \square

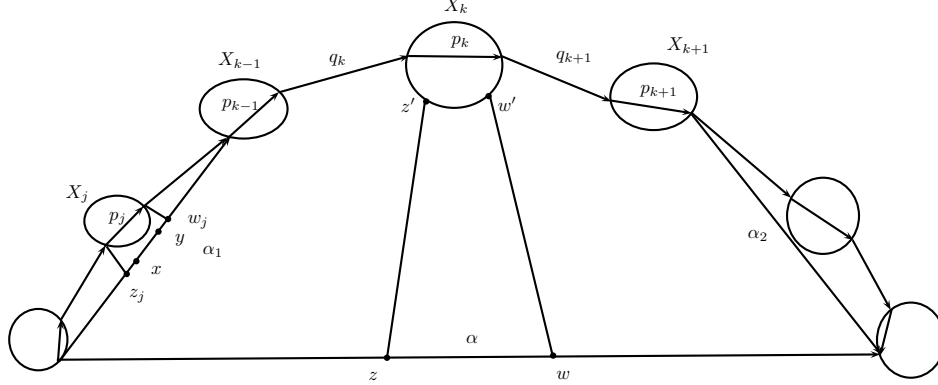


FIGURE 1. Proof of Proposition 2.16

A key step in proving Proposition 2.16 is the following.

Lemma 2.21. *Let X_k be a contracting subset for γ , where $0 < k < n$. Assume α is a geodesic such that $d(\gamma_-, \alpha_-) \leq \mu_{1,0}$, $d(\gamma_+, \alpha_+) \leq \mu_{1,0}$. Then there exist points $z, w \in \alpha \cap N_{\mu(1,0)}(X_k)$ such that $d(z, (p_k)_-) < R - 1$, $d(w, (p_k)_+) < R - 1$.*

Proof. Let $\alpha_1 = [\gamma_-, (p_{k-1})_-]$, $\alpha_2 = [(p_{k+1})_+, \gamma_+]$, $\beta_1 = [\gamma_-, (p_{k-1})_-]_\gamma$ and $\beta_2 = [(p_{k+1})_+, \gamma_+]_\gamma$. Note that $\alpha_1, \alpha_2, \beta_1, \beta_2$ may be trivial.

Apply Induction Assumption to β_1 . It follows that α_1 is a R -fellow traveller for β_1 . Let $z_j, w_j \in \alpha_1$ be the points given by Definition 2.15. Note that $d(z_j, w_j) \geq 1$ and $d((p_j)_-, z_j) < R$, $d((p_j)_+, w_j) < R$.

Let x, y be the first and last points on α_1 respectively such that $x, y \in N_{\mu(1,0)}(X_k)$. See Figure 1.

Claim. *The segment $[x, y]_{\alpha_1}$ contains no points from $\{z_j, w_j : 1 < j < k-1\}$. Moreover, any geodesic segment $[(p_j)_-, z_j]$ and $[(p_j)_+, w_j]$ have at least a distance $\mu_{1,0}$ to X_k .*

Proof of Claim. Suppose, to the contrary, that there is a point, say z_j from $\{z_j, w_j : 1 < j < k-1\}$ such that $z_j \in [x, y]_{\alpha_1}$. Note that any point on $[x, y]_{\alpha_1}$ has at most a distance $\sigma_{\mu(1,0)}$ to X_k . As $d(z_j, (p_j)_-) < R$, we have $d((p_j)_-, X_k) < R + \sigma_{\mu(1,0)}$. This contradicts Lemma 2.20, as it follows that β_1 has at least a distance $(R + \sigma_{\mu(1,0)})$ to X_k .

By the same argument, one can see that any geodesic segment $[(p_j)_-, z_j]$ and $[(p_j)_+, w_j]$ have at least a distance $\mu_{1,0}$ to X_k . \square

By the **Claim** above, we assume that $x, y \in [z_j, w_j]_\alpha$ for some $z_j, w_j \in \{z_j, w_j : 1 < j < k-1\}$. (The case that $x, y \in [w_j, z_{j+1}]_\alpha$ is similar).

Using projection, we shall show that $\alpha \cap N_{\mu(1,0)}(X_k) \neq \emptyset$. Suppose not. The length of p_k is estimated as follows:

$$\begin{aligned}
 \|\Pi_k([\gamma_-, (p_k)_-]_\gamma)\| &< \|\Pi_k([\alpha_-, z_j]_{\alpha_1})\| + \|\Pi_k([(p_j)_-, z_j])\| \\
 &+ \|\Pi_k(p_j)\| + \|\Pi_k([(p_j)_+, w_j])\| \\
 &+ \|\Pi_k([w_j, (\alpha_1)_+]_{\alpha_1})\| + \|\Pi_k(p_{k-1}q_k)\| \\
 &< 5\epsilon_{1,0} + B_{\lambda,c}.
 \end{aligned} \tag{10}$$

Similarly, we obtain that

$$\|\Pi_k([(p_k)_+, \gamma_+]_\gamma)\| < 5\epsilon_{1,0} + B_{\lambda,c}. \tag{11}$$

Note that $[\alpha_-, \gamma_-]$ and $[\alpha_+, \gamma_+]$ are of length at most $\mu_{1,0}$. Hence we have

$$\|\Pi_k([\alpha_-, \gamma_-])\| \leq \epsilon + 3\mu_{1,0}, \quad \|\Pi_k([\alpha_+, \gamma_+])\| \leq \epsilon + 3\mu_{1,0}. \tag{12}$$

It follows from (10), (12) and (11) that

$$\begin{aligned}
 \ell(p_k) &< \|\Pi_k([\gamma_-, (p_k)_-]_\gamma)\| + \|\Pi_k([(p_k)_+, \gamma_+]_\gamma)\| + \|\Pi_k(\alpha)\| \\
 &+ \|\Pi_k([\alpha_-, \gamma_-])\| + \|\Pi_k([\alpha_+, \gamma_+])\| \\
 &< 13\epsilon_{1,0} + 6\mu_{1,0} + 2B_{\lambda,c}.
 \end{aligned}$$

This gives a contradiction with $\ell(p_k) > D$, as it is assumed that

$$D > 13\epsilon_{1,0} + 6\mu_{1,0} + 2B_{\lambda,c}. \tag{13}$$

Hence $\alpha \cap N_{\mu(1,0)}(X_k) \neq \emptyset$. Let z be the first point of α such that $d(z, X_k) \leq \mu_{1,0}$. Let z' be a projection point of z to X_k . Then it follows that

$$\begin{aligned}
 d(z, (p_k)_-) &< d(z, z') + d(z', (p_k)_-) \\
 &< \mu_{1,0} + 5\epsilon_{1,0} + B_{\lambda,c} < R - 1,
 \end{aligned}$$

as it is assumed that

$$R > \mu_{1,0} + 5\epsilon_{1,0} + B_{\lambda,c} + 1. \tag{14}$$

Let w be the last point of α such that $d(z, X_k) \leq \mu_{1,0}$. Arguing in the same way, we see that $d(w, (p_k)_+) < R - 1$. This completes the proof. \square

We now finish the proof of Proposition 2.16, which is repeated applications of Lemma 2.21.

Proof of Proposition 2.16. Recall that γ is a (D, λ, c) -admissible path with at most $(n + 1)$ contracting subsets. Let α be a geodesic such that $\alpha_- = \gamma_-$ and $\alpha_+ = \gamma_+$. By Lemma 2.19, we can assume that $n \geq 2$.

Consider a contracting subset X_k for γ , where $0 < k < n$. By Lemma 2.21, there exist $z_k, w_k \in \alpha \cap N_{\mu(1,0)}$ such that $d(z_k, (p_k)_-) < R - 1$, $d(w_k, (p_k)_+) < R - 1$.

Let w'_k be a projection point of w_k to X_k . Then $d(w_k, w'_k) \leq \mu_{1,0}$. We consider $j = k + 1$. Let $\gamma' = [w'_k, (p_k)_+][(p_k)_+, \gamma_+]_\gamma$ and $\alpha' = [w_k, \alpha_+]_\alpha$.

Observe that γ' is a (D, λ, c) -admissible path with at most n contracting subsets. Apply Lemma 2.21 to γ' and α' . Then there exist points $z_j, w_j \in \alpha' \cap N_{\mu(1,0)}(X_j)$ such that $d(z_j, (p_j)_-) < R - 1$, $d(w_j, (p_j)_+) < R - 1$.

Continuously increasing or decreasing k , the points z_j, w_j on α are obtained to satisfy $d(z_j, (p_j)_-) < R - 1$, $d(w_j, (p_j)_+) < R - 1$ for all $0 \leq j \leq n$.

The conclusion that α is a R -fellow traveller follows from a slight modification z_j, w_j such that $d(z_j, w_j) \geq 1$, $d(z_j, (p_j)_-) < R$, $d(w_j, (p_j)_+) < R$. The proof is now complete. \square

3. FULLY QUASICONVEX SUBGROUPS

In this section, a finitely generated group G is always assumed to be hyperbolic relative to $\mathbb{P} = \{P_i : 1 \leq i \leq n\}$. We refer the reader to [19], [11], [5], [10] and [21] for the references on the relative hyperbolicity of a group.

Given a finite generating set S , let $\mathcal{G}(G, S)$ be the Cayley graph of G with respect to S . We denote by $\mathbf{Lab}(\cdot)$ the label function assigning for a combinatorial path in $\mathcal{G}(G, S)$ the product of generators labeling its edges in G .

Let $Y = \mathcal{G}(G, S)$, d be the combinatorial metric induced on the graph $\mathcal{G}(G, S)$ and $\mathbb{X} = \{gP : g \in G, P \in \mathbb{P}\}$. The conjugate of a subgroup of $P \in \mathbb{P}$ is called a *parabolic subgroup*, a left P -coset a *parabolic coset*.

3.1. Relatively quasiconvex subgroups. Relative quasiconvexity of a subgroup has been extensively studied from different points of view. See [21] and [15] for the equivalence of various definitions.

In this subsection, we shall recall a definition of relatively quasiconvex subgroup in terms of the geometry of Cayley graphs. This definition rather replies on the fact that \mathbb{X} is a contracting system with bounded intersection in Y .

Lemma 3.1 (Bounded intersection). [10] *There exists a positive real-valued function $\nu : \mathbb{R} \rightarrow \mathbb{R}_+$ such that any two distinct $gP, g'P' \in \mathbb{X}$ have ν -bounded intersection.*

The notion of transition points was introduced by Hruska in [21] and further generalized by Gerasimov-Potyagailo in [15].

Definition 3.2. Let p be a path in $\mathcal{G}(G, S)$ and v a point in p . Given $U > 0, L > 0$, we say v is (U, L) -deep in some $X \in \mathbb{X}$ if $\|[v, p_-]_p \cap N_U(X)\| > L$ and $\|[v, p_+]_p \cap N_U(X)\| > L$. If v is not (U, L) -deep in any $X \in \mathbb{X}$, then v is called a (U, L) -transition point of p .

Remark 3.3. Let v be a (U, L) -deep point. If $L > \nu(U)$, then v is (U, L) -deep in a unique $X \in \mathbb{X}$ by Lemma 3.1.

The following lemma is clear by the ν -bounded intersection and Remark 3.3. It roughly says that the points at which a path exits a parabolic coset is transitional.

Lemma 3.4. *Given $X \in \mathbb{X}, U > 0$, let z, w be the first and last points on a path p in $\mathcal{G}(G, S)$ such that $d(z, X) \leq U, d(w, X) \leq U$. If $d(z, w) > \nu(U)$, then z, w are $(U, \nu(U))$ -transition points of p .*

In terms of transition points, we can state a weak Morse Lemma for a pair of quasigeodesics in the Cayley graph $\mathcal{G}(G, S)$. It was originally proved in [21] that the Hausdorff distance between the transition points of an (absolute) geodesic and vertices of a relative geodesic is bounded. The following general version essentially follows from [15, Proposition 5.2.3].

Lemma 3.5. *Given $\lambda \geq 1, c \geq 0$, there exists a constant $U = U(\lambda, c) > 0$ such that the following holds for any two (λ, c) -quasigeodesics p, q in $\mathcal{G}(G, S)$ with same endpoints.*

For any $U_0 \geq U, L > \nu(U_0)$, there is a constant $R = R(U_0, L) > 0$ such that any (U_0, L) -transition point of p has at most a distance R to a (U_0, L) -transition point of q .

Remark 3.6. Fix $U_0 \geq U(\lambda, c)$ and $L_1, L_2 > \nu(U_0)$. By Lemma 3.4, it is easy to see that the set of (U_0, L_1) -transition points of a (λ, c) -quasigeodesic has a bounded Hausdorff distance H to the set of its (U_0, L_2) -transition points, where H depends on L_1, L_2 only.

We are now ready to state the definition of relatively quasiconvex subgroups.

Definition 3.7 (Relative quasiconvexity). Let $U = U(\lambda, c), L = \nu(U) + 1$, where $U(\cdot, \cdot), \nu(\cdot)$ are given by Lemmas 3.1 and 3.5 respectively. A subgroup H of G is called *relatively M -quasiconvex* for some $M > 0$ if for any (λ, c) -quasigeodesic p in $\mathcal{G}(G, S)$ with endpoints in H , any (U, L) -transition point of p lies in M -neighborhood of H .

To close this subsection, we recall two results frequently used in next sections. The first result is a general fact about the intersection of two subgroups in a countable group.

Lemma 3.8. [21] [23] *Suppose H, K be subgroups of a countable group G . Let d be a left invariant proper metric on G . Then for any $H, gK, U > 0$, there exists a constant $\kappa = \kappa(H, gK, U)$ such that $N_U(H) \cap N_U(gK) \subset N_\kappa(H \cap K^g)$.*

The next result is well-known but we could not locate a reference in the literature. Hence a proof is given here for completeness.

Lemma 3.9 (Long parabolic intersection). *Suppose H is relatively quasiconvex in G . Given a constant $U > 0$, there exists a constant $L = L(H, U)$ such that if $\|N_U(gP) \cap H\| > L$ for some $g \in G, P \in \mathbb{P}$, then $|H \cap P^g| = \infty$.*

Proof. Given $U > 0$, let $B = \{g : d(1, g) \leq U\}$ and $A = \max\{\kappa(H, gP, U) : g \in B, P \in \mathbb{P}\}$, where κ is the function given by Lemma 3.8. Consider the finite collection $\mathbb{F} = \{H \cap P^g : |H \cap P^g| < \infty, g \in B, P \in \mathbb{P}\}$. Set $L = \max\{d(1, g) : g \in N_A(\cup_{F \in \mathbb{F}} F)\}$. We claim that L is the desired constant.

Let $h \in N_U(gP) \cap H$ and $p \in P$ such that $d(h, gp) < U$. Set $g_0 = h^{-1}gp$. Note that $\|N_U(gP) \cap H\| = \|N_U(g_0P) \cap H\|$, where $d(1, g_0) < U$. Hence by Lemma 3.8, $N_U(g_0P) \cap H \subset N_A(P^{g_0} \cap H)$. This implies that if $\|N_U(g_0P) \cap H\| > L$, then $P^{g_0} \cap H$ is infinite. Hence, $P^g \cap H$ is infinite. \square

3.2. Fully quasiconvex subgroups. The notion of a fully quasiconvex subgroup is the central object in next sections.

Definition 3.10 (Fully quasiconvex subgroups). Let H be relatively quasiconvex in G . Then H is said to be *fully quasiconvex* if $H \cap P^g$ is either finite or of finite index in P^g for each $g \in G, P \in \mathbb{P}$.

The fundamental fact in this study is that a fully quasiconvex subgroup is a contracting subset. In particular, a collection of left cosets of a fully quasiconvex subgroup is a contracting system which we will focus on in next sections.

Lemma 3.11. [15, Proposition 8.2.4] *Let H be fully quasiconvex in G . For any $\lambda \geq 1, c \geq 0$, there exist positive constants $\mu = \mu(\lambda, c)$ and $\epsilon = \epsilon(\lambda, c)$ such that for any (λ, c) -quasigeodesic γ satisfying $d(\gamma, H) \geq \mu(\lambda, c)$ in $\mathcal{G}(G, S)$, we have $\|\Pi_H \gamma\| < \epsilon(\lambda, c)$.*

Remark 3.12. Note that maximal parabolic subgroups are fully quasiconvex. Hence by Lemma 3.1, \mathbb{X} is a (ϵ, μ) -contracting system with ν -bounded intersection.

The following lemma is easy exercise by the definition of full quasiconvexity.

Lemma 3.13. *Suppose H is fully quasiconvex in G . Then there exists a constant $U = U(H)$ such that if $|H \cap P^g| = \infty$ for some $P \in \mathbb{P}, g \in G$, then $gP \subset N_U(H)$.*

We now come to the key fact that enables us to build a normal form in the combination theorem in Section 4.

Lemma 3.14 (Orthogonality). *Suppose H is relatively quasiconvex and K is fully quasiconvex in G . For any constant $U > 0$, there exists a constant $\tau = \tau(U) > 0$ such that the following statement is true.*

Let $h \in H$ be such that $d(1, h) = d(1, ChC)$, where $C = H \cap K$. Then for any geodesic q between 1 and h , we have $\|q \cap N_U(K)\| < \tau(U)$ and $\|q \cap N_U(hK)\| < \tau(U)$.

Proof. By assumption, we have that $d(1, h) = d(1, hC)$ and $d(1, h) = d(1, Ch)$. It is easy to see that $d(1, h) = d(1, hC)$ implies that $d(1, h^{-1}) = d(1, Ch^{-1})$. Observe that the verification of the inequality $\|q \cap N_U(hK)\| < \tau$ can be reduced to $\|h^{-1}q \cap N_U(K)\| < \tau$, where $h^{-1}q$ is a geodesic between h^{-1} and 1. Hence, it suffices to verify the following claim.

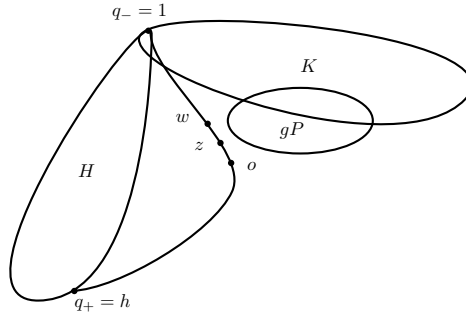


FIGURE 2. Proof of Lemma 3.14

□

Claim. *Let $d(1, h) = d(1, Ch)$. Then $\|q \cap N_U(K)\| < \tau$.*

Proof of Claim. Since H is relatively quasiconvex, there exist U_0, L_0, M such that any (U_0, L_0) -transition point of q lies in $N_M(H)$.

Given $U > 0$, let z be the last point on q such that $d(z, K) \leq U$. Since K is fully quasiconvex, we have $d(z, K) \leq \sigma(U)$, where σ is given by Lemma 2.4. See Figure 2.

Let w be the last (U_0, L_0) -transition point of q such that $w \in [1, z]_q$. Hence $d(w, H) \leq M$. By Lemma 3.8, there exists $w' \in H \cap K = C$ such that $d(w, w') \leq \kappa(H, K, M + \sigma(U))$. Since $d(1, h) = d(1, Ch)$, it follows that $d(1, w) \leq \kappa(H, K, M + \sigma(U))$. In fact, if $d(1, w) > d(w', w)$, then we obtain

$$d(1, h) = d(1, w) + d(w, h) > d(w', w) + d(w, h) \geq d(1, w'^{-1}h),$$

contradicting the choice of h .

Without loss of generality, we are going to bound $d(1, z)$ under the assumption that $d(w, z) > \max\{L(H, U), L(K, U)\}$. Note that $L(H, U), L(K, U)$ are given by Lemma 3.9.

Since w is the last (U_0, L_0) -transition point in $[1, z]_q$. Then z is (U_0, L_0) -deep in a unique $gP \in \mathbb{X}$. In particular, it follows that $w \in N_{U_0}(gP)$. Indeed, the first point of $[1, z]_q$ such that $z \in N_{U_0}(gP)$ is a (U_0, L_0) -transition point of q by Lemma 3.4.

Since $d(z, w) > L(K, U)$, we see that $|K \cap P^g| = \infty$. Then $gP \subset N_{U_1}(K)$, where $U_1 = U(K)$ is given by Lemma 3.13.

Let o be the last point on $[z, q_+]_q$ such that $o \in N_{U_0}(gP)$. Then o has to be a (U_0, L_0) -transition point by Lemma 3.4. By the relative quasiconvexity of H , we have $d(o, H) \leq M$. Since $d(o, w) > d(z, w) > L(H, U)$, it follows by Lemma 3.9 that $|H \cap P^g| = \infty$.

Note that $o \in N_M(H) \cap N_{U_0}(gP) \subset N_M(H) \cap N_{U_0+U_1}(K)$. By Lemma 3.8, there is $o' \in H \cap K = C$ such that $d(o, o') \leq \kappa(H, K, M + U_0 + U_1)$.

Since $d(1, h) = d(1, Ch)$, we obtain as above that $d(1, o) \leq d(o, o') \leq \kappa(H, K, M + U_0 + U_1)$. Hence $d(1, z) \leq d(1, o) \leq \kappa(H, K, M + U_0 + U_1)$, completing the proof of Claim. \square

4. COMBINING FULLY QUASICONVEX SUBGROUPS

4.1. The setup. Let H be relatively quasiconvex and K fully quasiconvex in a relatively hyperbolic group G . Denote $C = H \cap K$. Let $Y = \mathcal{G}(G, S)$ and $\mathbb{X} = \{gK : g \in G\}$. We choose \mathcal{L} to be the set of all quasigeodesics in Y .

Let μ, ϵ be the common functions given by Lemma 3.11 for fully quasiconvex subgroups K and all $P \in \mathbb{P}$. Then \mathbb{X} is a (μ, ϵ) -contracting system with respect to \mathcal{L} . (\mathbb{X} may not have bounded intersection property.)

Let σ be the quasiconvexity function given by Lemma 2.4. Then H, K and each $P \in \mathbb{P}$ are σ -quasiconvex.

4.2. Normal forms in $H \star_C K$. Let's consider the case $g = k_0 h_1 k_1 \dots h_n k_n$, where $h_i \in H \setminus C, k_i \in K \setminus C$. The other form of g is completely analogous.

For each $1 \leq i \leq n$, let $h_i \in H$ be such that $d(1, h_i) = d(1, Ch_i C)$. It is easy to see that such a representation of g always exists.

Let $\gamma = p_0 q_1 p_1 \dots q_n p_n$ be a concatenation of geodesic segments p_i, q_i in $\mathcal{G}(G, S)$ such that $(p_0)_- = 1$ and $\mathbf{Lab}(p_i) = k_i, \mathbf{Lab}(q_i) = h_i$. We call $\gamma = p_0 q_1 p_1 \dots q_n p_n$ a *normal path* of g .

Define $f_i = k_0 h_1 k_1 \dots k_{i-1} h_i$ for $1 \leq i \leq n$ and $f_0 = 1$. Then p_i has two endpoints in the left coset $f_i K \in \mathbb{X}$.

The following lemma is almost obvious by the definition of a normal path.

Lemma 4.1. *There exists a constant $D = D(H, K) > 0$, $\Lambda = \Lambda(H, K) > 0$ such that the following statement is true.*

Suppose that there are $\dot{H} \subset H$ and $\dot{K} \subset K$ such that $\dot{H} \cap \dot{K} = C$ and $d(1, g) > D$ for any $g \in (\dot{H} \cup \dot{K}) \setminus C$. Then the normal path of any element in $\dot{H} \star_C \dot{K}$ is a $(D, 1, 0)$ -admissible path and thus a $(\Lambda, 0)$ -quasigeodesic.

Proof. Let $D = D(H, K)$, $\Lambda = \Lambda(H, K)$ be the constants given by Main Corollary 2.17.

Let $g = k_0 h_1 k_1 \dots h_n k_n$ be an element in $\dot{H} \star_C \dot{K}$, where $k_i \in \dot{K} \setminus C$, $h_i \in \dot{H} \setminus C$. By assumption, it follows that $d(1, h_i) > D$, $d(1, k_i) > D$.

Let $\gamma = p_0 q_1 p_1 \dots q_n p_n$ be the associated normal path. Hence $\ell(p_i) > D$, $\ell(q_i) > D$. By Lemma 3.14, q_i is ν -orthogonal to $g_{i-1} K$, $g_i K$. Hence γ is a $(D, 1, 0)$ -admissible path. By Main Corollary 2.17, γ is a $(\Lambda, 0)$ -quasigeodesic. \square

Remark 4.2. Note that if K is a maximal parabolic subgroup, then it suffices to assume that $d(1, g) > D$ for any $g \in \dot{K} \setminus C$. This follows from the fact that $\{gP : g \in G, P \in \mathbb{P}\}$ has bounded intersection property. Hence, the second case of Condition (3) in the definition of admissible paths is always satisfied.

4.3. Proof of Theorem 1.1. Lemma 4.1 implies that $\dot{H} \star_C \dot{K} \rightarrow \langle \dot{H}, \dot{K} \rangle$ is injective.

To show the relative quasiconvexity of $\langle \dot{H}, \dot{K} \rangle$, note that the normal path of each element in $\langle \dot{H}, \dot{K} \rangle$ is a $(\Lambda, 0)$ -quasigeodesic. Let $U = U(\Lambda, 0)$ be the constant given by Lemma 3.5 and $L = \nu(U) + 1$. Observe that any (U, L) -transition point of γ is a (U, L) -transition point of either p_i or q_i . Since H, K are relatively quasiconvex, we see that any (U, L) -transition point of γ has a uniform bounded distance to $\langle \dot{H}, \dot{K} \rangle$. Hence by Lemma 3.5, $\langle \dot{H}, \dot{K} \rangle$ is relatively quasiconvex.

We now show the last statement of Theorem 1.1 about the conjugacy classes of parabolic subgroups.

Lemma 4.3. *Every parabolic subgroup of $\langle \dot{H}, \dot{K} \rangle$ is conjugated into either \dot{H} or \dot{K} .*

Proof. Note that maximal parabolic subgroups in $\langle \dot{H}, \dot{K} \rangle$ are of form $P^f \cap \langle \dot{H}, \dot{K} \rangle$, where $f \in G$ and $P \in \mathbb{P}$. Let $g \in \langle \dot{H}, \dot{K} \rangle \setminus (\dot{H} \cup \dot{K})$. The idea is to take sufficiently large D in Theorem 1.1, to show that $g \notin P^f$ for any $f \in G, P \in \mathbb{P}$.

Suppose, to the contrary, that $g = f p f^{-1}$ for some $f \in G, p \in P$. Without loss of generality, we assume that $g = h_0 k_1 h_1 \dots k_n h_n$, where $h_i \in \dot{H}, k_i \in \dot{K}$. Denote the normal path of g by $\gamma = p_0 q_1 p_1 \dots q_n p_n$.

Let α be a geodesic segment with the same endpoints as γ . By Proposition 2.16, the endpoints of each p_i, q_i lie in a uniform R -neighborhood of α , where $R = R(1, 0)$.

Let z, w be the first and last points of α respectively such that $z, w \in N_{\mu(1,0)}(fP)$. Then by projection, we have $d(\alpha_-, z), d(\alpha_+, w) \leq d(1, f) + \mu_{1,0} + \epsilon_{1,0}$. Let $\alpha' = [z, w]_\alpha$. Then $\alpha' \subset N_{\sigma(\mu(1,0))}(fP)$.

Note that the analysis in the last paragraph applies to any power of g . By taking sufficiently large power of g , the length of α' can be arbitrarily large.

Set $U = \sigma(R + \sigma(\mu_{1,0}))$. Hence we can assume further that there exist consecutive p_i, q_i for some i of γ such that $q_i, p_i \subset N_U(fP)$. Let $f_i \in G$ be the element associated to the vertex $(q_i)_+ = (p_i)_-$. Then q_i, p_i have endpoints in $f_i H, f_i K$ respectively.

Since $\|N_U(fP) \cap f_i H\| > D$ and we assumed that $D > L(H, U)$, it follows by Lemma 3.9 that $H \cap P^{f'}$ is infinite, where $f' = f_i^{-1}f$. Similarly, as it is assumed that $D > L(K, U)$, we have $K \cap P^{f'}$ is infinite. By Lemma 3.13, we see that $f'P \subset N_U(K)$.

We now translate the terminal point of q_i to 1. Note that $f_i^{-1}q_i \subset N_U(f'P)$. Since $k_i \in K$ is such that $d(1, k_i) = d(1, Ck_iC)$. By Lemma 3.14, we have $\|f_i^{-1}q_i \cap N_U(K)\| < \tau(U)$. This implies that $\ell(q_i) < \tau(U)$. It suffices to assume further $D > \tau(U)$ to get a contradiction. Hence, it is shown that any $g \notin P^f$ for $f \in G, P \in \mathbb{P}$. \square

In fact, the proof also shows the following.

Corollary 4.4. *Any element g in $\langle \dot{H}, \dot{K} \rangle \setminus (\dot{H} \cup \dot{K})$ is hyperbolic, i.e.: g is not conjugated into any $P \in \mathbb{P}$.*

We also note the following corollary.

Corollary 4.5. *The virtual amalgamation of two fully quasiconvex subgroups is fully quasiconvex.*

5. HNN COMBINATION THEOREM

As in the previous section, a finitely generated group G is assumed to be hyperbolic relative to a collection of subgroups \mathbb{P} . In addition, we will also consider the geometry of *relative Cayley graph* of G with respect to \mathbb{P} , denoted by $\mathcal{G}(G, S \cup \mathbb{P})$.

Note that $\mathcal{G}(G, S)$ is a subgraph of $\mathcal{G}(G, S \cup \mathbb{P})$, and a path (resp. geodesic) in $\mathcal{G}(G, S \cup \mathbb{P})$ is also referred to as a *relative path* (resp. *relative geodesic*).

5.1. The setup. Let $Y = \mathcal{G}(G, S)$ and $\mathbb{X} = \{gP : g \in G, P \in \mathbb{P}\}$. Let μ, ϵ be the common functions given by Lemma 3.11 for all $P \in \mathbb{P}$. We choose \mathcal{L} to be the set of all quasigeodesics in Y . Then \mathbb{X} is a (μ, ϵ) -contracting system with respect to \mathcal{L} .

Let σ be the quasiconvexity function given by Lemma 2.4. Then each $P \in \mathbb{P}$ are σ -quasiconvex.

5.2. Lift paths. The notion of a lift path is interacting between the geometry of relative and normal Cayley graphs. Before giving the definition, we need recall several notions introduced by Osin [28] in relative Cayley graphs.

Definition 5.1 (P_i -components). Let γ be a path in $\mathcal{G}(G, S \cup \mathbb{P})$. Given $P_i \in \mathbb{P}$, a subpath p of γ is called *P_i -component* if $\mathbf{Lab}(p) \in P_i$ and no subpath q of γ exists such that $p \subsetneq q$ and $\mathbf{Lab}(q) \in P_i$.

Two P_i -components p_1, p_2 of γ are *connected* if $(p_1)_-, (p_2)_-$ belong to a same $X \in \mathbb{X}$. A P_i -component p is *isolated* if no other P_i -component of γ is connected to p .

Definition 5.2 (Lift path). Let γ be a path in $\mathcal{G}(G, S \cup \mathbb{P})$. The *lift path* $\hat{\gamma}$ is obtained by replacing each P_i -component of γ by a geodesic with the same endpoints in $\mathcal{G}(G, S)$.

We recall a fact implicitly in [10, Thm. 1.12(4)] that the lift path of a relative geodesic is a quasigeodesic. A rather general version can be found in [15, Proposition 7.2.2].

Lemma 5.3. *There exists a constant $\lambda \geq 1$ such that the lift of any relative geodesic is a $(\lambda, 0)$ -quasigeodesic.*

The following result says that a relative geodesic leaves parabolic cosets in an orthogonal way, as is defined in Section 2.

Lemma 5.4 (Orthogonality of relative geodesics). *For any constant $U > 0$, there exists a constant $\tau = \tau(U) > 0$ such that the following holds.*

Given $gP \in \mathbb{X}$, let p be a relative geodesic such that $p \cap gP = \{p_+\}$. Denote by \hat{p} the lift of p . Then $\|\hat{p} \cap N_U(gP)\| < \tau$.

Proof. By Lemma 2.10, $\mathbb{X} = \{gP : g \in G, P \in \mathbb{P}\}$ has B -bounded projection for some $B > 0$. Moreover, the projection of an edge in $\mathcal{G}(G, S)$ to any $X \in \mathbb{X}$ is also uniformly bounded by a constant, say B for convenience.

Given $U > 0$, set $\nu(U) = 4B(U+1)^2 + 2(U+1)$. Let z be the first point of \hat{p} such that $d(z, gP) \leq U$. We shall show that $d(z, p_+) < \nu(U)$.

Without loss of generality, assume that z is a vertex in a geodesic segment \hat{s} , which is the lift of a P_i -component s of p . Clearly $d_{S \cup \mathcal{P}}(z, s_+) \leq 1$. Let $w \in gP$ such that $d_S(z, w) \leq U$.

Let q be a geodesic in $\mathcal{G}(G, S)$ such that $q_- = z, q_+ = w$. Since $d_{S \cup \mathcal{P}}(w, p_+) \leq 1$, let e_1 be an edge such that $\mathbf{Lab}(e_1) \in P$ and $(e_1)_- = w, (e_1)_+ = p_+$. Similarly, let e_2 be an edge such that $\mathbf{Lab}(e_2) \in P_i$ and $(e_2)_- = s_+, (e_2)_+ = z$. Consider the cycle $o = qe_1[p_+, s_+]_Pe_2$. Note that

$$\ell(o) \leq \ell(q) + 1 + d_{S \cup \mathcal{P}}(s_+, p_+) + 1 \leq 2(U+1).$$

Since p is a relative geodesic, each P_i -component of o is isolated. Then given a P_i -component t of o , we project other edges of o to the parabolic coset associated to t . This gives the estimate $d(t_-, t_+) \leq B\ell(o)$ for each P_i -component t of o . Hence, it follows that $d(z, p_+) \leq \ell([z, p_+]_{\hat{p}}) \leq \ell(o) + B(\ell(o))^2 < 4B(U+1)^2 + 2(U+1)$. \square

We now consider a class of admissible paths coming from the lifts of piecewise relative geodesics. Such type of admissible paths will be obtained by truncating the normal path defined in the next subsection.

Lemma 5.5. *There are constants $D > 0, \Lambda \geq 1$ such that the following holds.*

Let $\gamma = p_0q_1p_1 \dots q_np_n$ be a concatenation path in $\mathcal{G}(G, S \cup \mathcal{P})$, where p_i are P_i -components of γ for some $P_i \in \mathbb{P}$ and q_i are relative geodesics. Assume that $d((p_i)_-, (p_i)_+) > D$ for $0 < i < n$, and p_{i-1}, p_i are not connected. Then the lift of γ is a $(\Lambda, 0)$ -quasigeodesic.

Proof. Let $\hat{\gamma} = \hat{p}_0\hat{q}_1\hat{p}_1 \dots \hat{q}_n\hat{p}_n$ be the lift path. Each \hat{q}_i is a $(\lambda, 0)$ -quasigeodesic for some $\lambda \geq 1$ by Lemma 5.3. Let $g_iP_i \in \mathbb{X}$ be the parabolic coset in which the endpoints of p_i lie. Note that Lemma 5.4 verifies that \hat{q}_i is orthogonal to $g_{i-1}P_{i-1}, g_iP_i$. Hence, we see that $\hat{\gamma}$ is a $(D, \lambda, 0)$ -admissible path. Since \mathbb{X} is a contracting system. As a consequence, the constants D, Λ are provided by Main Corollary 2.17. \square

5.3. Normal forms in $H \star_{Q^t=Q'}$. Let $P \in \mathbb{P}, f \in G$ be such that $Q = P \cap H$ and $Q^f = Q'$ are non-conjugate maximal parabolic subgroups of H . Denote $P' = P^f$.

Assume that there is $c \in P$ such that $Q^c = Q$. Set $t = fc$.

Let $g \in H \star_{Q^t=Q'}$ be written as the form $h_1 t^{\epsilon_1} h_2 t^{\epsilon_2} \dots h_n t^{\epsilon_n}$, where $h_i \in H, \epsilon_i \in \{1, -1\}$. By Britton's Lemma, if $\epsilon_i = 1, \epsilon_{i+1} = -1$, then $t \notin Q$; if $\epsilon_i = -1, \epsilon_{i+1} = 1$, then $t \notin Q'$.

A *normal path* of g is a concatenated path $\gamma = q_1(\beta_1 p_1)^{\epsilon_1} q_2(\beta_2 p_2)^{\epsilon_2} \dots q_n(\beta_n p_n)^{\epsilon_n}$ in $\mathcal{G}(G, S \cup \mathcal{P})$ with the following properties

- (1) q_i is a relative geodesic in $\mathcal{G}(G, S \cup \mathcal{P})$ such that $\mathbf{Lab}(q_i) = h_i$,
- (2) β_i is a geodesic in $\mathcal{G}(G, S)$ such that $\mathbf{Lab}(\beta_i) = f$, and
- (3) p_i is an edge in $\mathcal{G}(G, S \cup \mathcal{P})$ such that $\mathbf{Lab}(p_i) = c$.

Let $g_i P \in \mathbb{X}$ be the parabolic coset such that $(p_i)_-, (p_i)_+ \in g_i P$. These $g_i P$ will serve as contracting subsets for a admissible path that we will construct. We shall first verify that consecutive $g_i P$ are distinct.

Lemma 5.6. *Peripheral cosets $g_{i-1} P, g_i P$ are distinct.*

Proof. In the following, we only verify the case $i = 1$. The other cases are completely analogous.

If $\epsilon_1 = 1, \epsilon_2 = -1$, then $\gamma = q_1(\beta_1 p_1) q_2(p_2^{-1} \beta_2^{-1}) \dots q_n(\beta_n p_n)^{\epsilon_n}$. It follows from Britton's Lemma that $\mathbf{Lab}(q_2) \notin Q$. Hence we see that $g_1 P, g_2 P$ are distinct.

If $\epsilon_1 = 1, \epsilon_2 = 1$, then $\gamma = q_1(\beta_1 p_1) q_2(\beta_2 p_2) \dots q_n(\beta_n p_n)^{\epsilon_n}$. Assume that $(p_1)_- = 1$. Let $g_2 P$ be the parabolic coset in which p_2 lies. Suppose, to the contrary, that $P = g_2 P$, that is $h_2 f \in P$. By assumption, note that $P^f = P'$ and thus $h_2^{-1} P h_2 = P'$. It follows that $h_2^{-1} Q h_2 = Q'$, contradicting the assumption that Q, Q' are not conjugate in H . \square

Note that the normal path is defined in $\mathcal{G}(G, S \cup \mathcal{P})$. So our next step is, before lifting each p_i, q_i , to truncate the extra part of γ lying inside $g_i P$ as follows.

Truncating the path γ . Given $g \in \langle H, t \rangle$, let

$$\gamma = q_1(\beta_1 p_1)^{\epsilon_1} q_2(\beta_2 p_2)^{\epsilon_2} \dots q_n(\beta_n p_n)^{\epsilon_n}$$

be its normal path. For each $g_i P$, if $q_i \cap g_i P \neq \emptyset$, then let z_i be the first point of q_i such that $z_i \in g_i P$; otherwise, let $z_i = (p_i^{\epsilon_i})_-$. In a similar way, if $q_{i+1} \cap g_i P \neq \emptyset$, then let w_i be the last point of q_{i+1} such that $w_i \in g_i P$; otherwise, let $w_i = (p_i^{\epsilon_i})_+$. Denote $w_0 = \gamma_-$.

Let q'_i be the lift path of the segment $[w_{i-1}, z_i]_\gamma$. Let p'_i be a geodesic in $\mathcal{G}(G, S)$ between z_i and w_i . Then $\bar{\gamma} = q'_1 p'_1 \dots q'_n p'_n$ is called the *truncation* of γ .

We now carefully examine the truncation paths and show that they are admissible paths. Let $|f| = d(1, f)$.

Lemma 5.7. *Given $D > 0$, assume that $d(1, g) > D$ for any $g \in cQ$. Let $D' = D - \kappa(H, f^{-1}P, M) - \kappa(H, P, M)$. Then the truncation path of any element in $\langle H, t \rangle \setminus H$ is a $(D', \lambda, (\lambda + 2)|f|)$ -admissible path.*

Recall that the number λ above is given by Lemma 5.3 and the function $\kappa(\cdot, \cdot, \cdot)$ given by Lemma 3.8.

Proof. Let $\gamma = q_1(\beta_1 p_1)^{\epsilon_1} q_2(\beta_2 p_2)^{\epsilon_2} \dots q_n(\beta_n p_n)^{\epsilon_n}$ be the normal path of an element in $\langle H, t \rangle$. Without loss of generality, we consider the case that $\epsilon_1 = 1$. The case that $\epsilon_1 = -1$ is symmetric by reversing the orientation of γ .

Let $(p_1)_- = 1$. Let z be the first vertex of q_1 such that $z \in P$ if it exists; otherwise let $z = (\beta_1)_+$. The relative M -quasiconvexity of H implies that $z \in N_M(f^{-1}H) \cap P$. By Lemma 3.8, there is $z' \in f^{-1}Hf \cap P = f^{-1}Q'f = Q$ such that $d(z', z) \leq \kappa(H, f^{-1}P, M)$.

Let w be the last vertex of q_2 such that $w \in P$. The relative M -quasiconvexity of H implies that $w \in N_M(cH) \cap P$. By Lemma 3.8, there is $w' \in H \cap P = Q$ such that $d(cw', w) \leq \kappa(H, P, M)$. See Figure 3.

Let q'_1 be the lift of the relative path $[\gamma_-, z]_\gamma$, and p'_1 a geodesic between z and w . By Lemma 5.3, we see that q'_1 is a $(\lambda, (\lambda + 2)|f|)$ -quasigeodesic.

Since $z'^{-1}cw' \in QcQ = cQ$, it follows that $d(z', cw') > D$. Then we have

$$(15) \quad \begin{aligned} \ell(p'_1) = d(z, w) &> d(z', cw') - d(z, z') - d(w, cw') \\ &> D - \kappa(H, f^{-1}P, M) - \kappa(H, P, M). \end{aligned}$$

We continuously truncate q_i to define q'_i, p'_i as above. Let $\bar{\gamma} = q'_1 p'_1 \dots q'_n p'_n$ be the truncation path, where p'_i are P -components. Moreover, the paths q'_i are $(\lambda, (\lambda + 2)|f|)$ -quasigeodesics. By Lemma 5.6, we see that $\bar{\gamma}$ is a $(D', \lambda, (\lambda + 2)|f|)$ -admissible path. \square

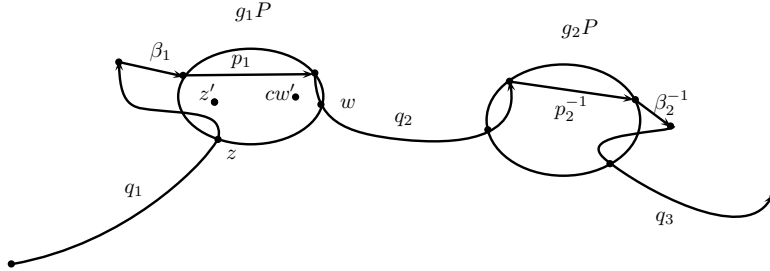


FIGURE 3. Proof of Lemma 5.7

5.4. Proof of Theorem 1.3. Let $D = D(\lambda, (\lambda + 2)|f|)$, $\Lambda = \Lambda(\lambda, (\lambda + 2)|f|)$ be the constants given by Corollary 2.17. Define

$$D(H, P, f) = D + 1 - \kappa(H, f^{-1}P, M) - \kappa(H, P, M).$$

By Lemma 5.7, the truncation path of any element in $H \star_{Q^t=Q'}$ is a $(\Lambda, 0)$ -quasigeodesic in $\mathcal{G}(G, S)$. Then $H \star_{Q^t=Q'} \rightarrow \langle H, t \rangle$ is injective.

We shall now show that $\langle H, t \rangle$ is relatively quasiconvex. Let $\bar{\gamma} = q'_1 p'_1 \dots q'_n p'_n$ be a truncation path obtained as above for an element in $\langle H, t \rangle \setminus H$. Let $U = U(\Lambda, 0)$ be the constant given by Lemma 3.5 and $L = \nu(U) + 1$. Observe that any (U, L) -transition point of $\bar{\gamma}$ is a (U, L) -transition point of either q'_i or p'_i .

By the definition of truncation, each q'_i is the lift of one of the following types of relative paths: 1). $\beta_{i-1}^{-1} q_i \beta_i$, 2). a subpath of either $\beta_{i-1}^{-1} q_i$, $q_i \beta_i$ or q_i . Note that q_i has two endpoints in a left H -coset and β_i is of the fixed length $|f|$. Then any (U, L) -transition point of q'_i lies in a uniform neighborhood of the associated left H -coset in all cases.

We now consider the subpath p'_i . By the analysis in the proof of Lemma 5.7 and the inequality (15) therein, we see that the endpoints of p'_i are at most a

distance $\kappa(H, f^{-1}P, M) - \kappa(H, P, M)$ to the endpoints of a geodesic $[z', cw']$ with label $z'^{-1}cw' \in QcQ = Qc$. Since $Q \subset H$ and $z' \in Q$, any (U, L) -transition point of $[z', cw']$ lies in a uniform neighborhood of the associated left Q -coset. Note that c is a fixed element. Then any (U, L) -transition point of p'_i has a uniformly bounded distance to a (U, L) -transition point of $[z', cw']$. Consequently, any (U, L) -transition point of p'_i lies in a uniform neighborhood of the associated left H -coset.

Therefore, it is verified that any (U, L) -transition point of $\bar{\gamma}$ lies in a uniform neighborhood of $\langle H, t \rangle$. This shows that the relative quasiconvexity of $\langle H, t \rangle$.

We now show the second statement of Theorem 1.3.

Lemma 5.8. *Every parabolic subgroup in $\langle H, t \rangle$ is conjugate into H .*

Sketch of Proof. Let $g \in \langle H, t \rangle \setminus H$. Similarly as Lemma 4.3, the idea is to take sufficiently large D in Theorem 1.3, to show that $g \notin P^f$ for any $f \in G$ and $P \in \mathbb{P}$.

Suppose, to the contrary, that $g = fpf^{-1}$, where $p \in P$. Let $\bar{\gamma} = q_1p_1q_2p_2 \dots q_n p_n$ be the truncation path of g , where p_i are P -components.

Let α be a geodesic segment with the same endpoints as γ . By Proposition 2.16, the endpoints of each p_i lie in a uniform R -neighborhood of α , where $R = R(1, 0)$.

Set $U := \sigma(R + \sigma(\mu_{1,0}))$ as in the proof of Lemma 4.3. By taking a sufficiently large power of g , we can assume further that there exist p_{i-1}, p_i of $\bar{\gamma}$ such that $p_{i-1}, p_i \subset N_U(fP)$.

Note that each p_i is a P -component with endpoints in some parabolic coset g_iP . By the ν -bounded intersection of \mathbb{X} and $D > \nu(U)$, we obtain that $g_iP = fP$. However, $g_{i-1}P, g_iP$ are distinct by Lemma 5.6. This gives a contradiction. Hence, it is shown that $g \notin P^f$ for any $f \in G, P \in \mathbb{P}$. \square

5.5. Proof of Corollary 1.5. Let $\{Q_1, \dots, Q_m\}$ be the conjugacy classes in H representing boundary components of a compact surface S . As H has no accidental parabolics in G , there exists parabolic subgroups P_1, \dots, P_m of G such that H is relatively quasiconvex in G and $Q_i = H \cap P_i$ are parabolic subgroups in H .

By Theorem 1.3 in [23], there exists a constant $D_1 = D(H, P_1)$ such that the following holds. Let pQ_1 be such that $\forall g \in pQ_1, d(1, g) > D_1$. Then we have that $H \star_{Q_1} H^p = \langle H, H^p \rangle$ is relatively quasiconvex in G .

Note that Q_i are all cyclic and P_i are of rank at least two. Then there exists $p_1 \in P_1$ such that any elements in p_1Q_1 has length bigger than D_1 . This implies that $H_1^+ = H \star_{Q_1} H^{p_1} = \langle H, H^{p_1} \rangle$ is relatively quasiconvex. Moreover, the complete set of conjugacy classes of parabolic subgroups in H_1^+ is $\{Q_1, Q_2, Q_2^{p_1}, \dots, Q_m, Q_m^{p_1}\}$.

Apply Theorem 1.3 to H_1^+, P_2 and p_1 . Let $D_2 = D(H_1^+, P_2, p_1)$ be the constant given by Theorem 1.3. As Q_2 is of infinite index in P_2 , there is an element $p_2 \in P_2$ such that any element in p_2Q_2 is of length bigger than D_2 . Let $t = p_1p_2$. It follows that $H_2^+ = \langle H_1^+, t \rangle$ is relatively quasiconvex. Moreover, the complete set of conjugacy classes of parabolic subgroups in H_2^+ are $\{Q_1, Q_2, \dots, Q_m, Q_m^{p_m}\}$.

After a finitely many steps, we obtain that H_m^+ is relatively quasiconvex and its conjugacy classes of parabolic subgroups are $\{Q_1, Q_2, \dots, Q_m\}$. On the other hand, one sees that H_m^+ is isomorphic a closed surface group. The proof is complete.

Remark 5.9. Note that the constructed surface subgroup H_m^+ has accidental parabolics in G .

6. SEPARABILITY OF DOUBLE COSETS

Suppose G is hyperbolic relative to a collection of slender LERF groups. Note that every subgroup of a slender LERF group is separable.

The following result is shown in [22] by a much involved proof. We here provide simpler proof using normal paths constructed in Section 3.

Lemma 6.1. *Suppose G is hyperbolic relative to a collection of slender LERF groups. Let H be relatively quasiconvex in G and an element $g \in G \setminus H$. Then there exists a fully quasiconvex subgroup H^+ such that $H \subset H^+$ and $g \notin H^+$.*

Proof. Let $g \in G \setminus H$. Let $P \in \mathbb{P}^G$ such that $|H \cap P| = \infty$ and $[P : H \cap P] < \infty$. Since $H \cap P$ is separable in P , we can combine H with a finite index subgroup $\dot{P} \subset P$, where each element in $\dot{P} \setminus H$ has sufficiently large word length. Thus by Lemma 4.1, so does any element in $\langle H, \dot{P} \rangle \setminus (H \cap P)$. As a consequence, this implies that $g \notin \langle H, \dot{P} \rangle$ (otherwise, it would lead to a contradiction that $g \in H \cap P$). After finitely many steps, we can get a fully quasiconvex subgroup containing H but avoiding g . \square

Corollary 6.2. *Under the assumption of Lemma 6.1, if G is separable on fully quasiconvex subgroups, then every relatively quasiconvex subgroup is separable.*

Suppose H is relatively quasiconvex and K fully quasiconvex in G . Let $C = H \cap K$.

Proof of Theorem 1.8. Since H', K' are separable, $C' = H' \cap K'$ is separable. Given any $g \notin H'K'$, it suffices to find a closed set separating $H'K'$ and g .

We first consider the case that $C' = C$.

Let $m = d(1, g)$. Let $\Lambda = \Lambda(H, K)$, $D = D(H, K)$ be the constants given by Lemma 4.1, and $D_1 = \max(m\Lambda, D)$. By the separability of C , there exists finite index subgroups \dot{H}, \dot{K} of H', K' respectively such that $d(1, f) > D_1$ for any $f \in \dot{H} \cup \dot{K} \setminus C$. By Theorem 1.1, $\langle \dot{H}, \dot{K} \rangle$ is relatively quasiconvex. Thus $\langle \dot{H}, \dot{K} \rangle$ is separable by Corollary 6.2.

We now claim that $g \notin H'\langle \dot{H}, \dot{K} \rangle K'$. Argue by way of contradiction. Suppose that $g \in H'\langle \dot{H}, \dot{K} \rangle K'$. Then there exists $h \in H', k \in K'$ such that $h g k \in \langle \dot{H}, \dot{K} \rangle$. Since $g \notin H'K'$, it follows that $h g k$ cannot be written as $h'k'$, where $h' \in \dot{H}, k' \in \dot{K}$. Hence, g has the following form $g = h_0 k_1 h_1 \dots k_n h_n k_{n+1}$ for $n \geq 1$, where $h_0 \in H', k_{n+1} \in K', h_i \in \dot{H} \setminus C, k_i \in \dot{K} \setminus C$. Let γ be the normal path of g . Since γ is a $(\Lambda, 0)$ -quasigeodesic, we have $d(1, g) > \Lambda^{-1} D_1 > m$. This is a contradiction. It follows that $g \notin H'\langle \dot{H}, \dot{K} \rangle K'$.

Since \dot{H}, \dot{K} are of finite index in H', K' respectively, there exists finitely many $h_i \in H', k_j \in K'$ such that $\cup h_i \dot{H} = H', \cup k_j \dot{K} = K'$. Observe that $H'K'$ is contained in $H'\langle \dot{H}, \dot{K} \rangle K' = \bigsqcup h_i \langle \dot{H}, \dot{K} \rangle k_j$, which is a finite union of closed sets. This shows that $H'K'$ is separable.

Let's now turn to the general case that C' is of finite index in C .

Denote by $\{1, c_1, c_2, \dots, c_n\}$ the set of left coset representatives of C' in C . In virtue of separability of H', K' , it is easy to see that there are two finite index subgroups \hat{H}, \hat{K} in H, K respectively such that $\hat{H} \cap \hat{K} = C'$. Note that \hat{K} fully quasiconvex. Applying the special case to $H_1 = H' \cap \hat{H}, K_1 = K' \cap \hat{K}$, we see $H_1 K_1$ is separable. Since H_1, K_1 are of finite index in H', K' respectively, it follows that $H'K'$ is separable. \square

Proof of Corollary 1.10. Let H, K be two parabolic subgroups. If H, K lie in different maximal parabolic subgroups, then $H \cap K$ is finite. The conclusion follows from Theorem 1.8. Now let H, K be in the same $P \in \mathbb{P}$. Since P is virtually abelian, it follows that the double coset of any two subgroups in P is separable in P and thus in G .

Note that by a result of Osin [27], any hyperbolic element g in G is contained in a virtually cyclic subgroup $E(g)$ such that G is hyperbolic relative to $\mathbb{P} \cup \{E(g)\}$. Hence it follows by the same argument as above that the double coset of any two cyclic subgroups is separable. \square

REFERENCES

1. E. Alibegovic, *A combination theorem for relatively hyperbolic groups*, Bull. London Math. Soc. (2005), no. 3, 459–466.
2. M. Baker and D. Cooper, *A combination theorem for convex hyperbolic manifolds, with applications to surfaces in 3-manifolds*, J. Topol. **1** (2008), no. 3, 603–642.
3. M. Bestvina and M. Feighn, *A combination theorem for negatively curved groups*, J. Diff. Geom. (1992), 85–101.
4. M. Bestvina and K. Fujiwara, *A characterization of higher rank symmetric spaces via bounded cohomology*, (2007), arXiv:math/0702274.
5. B. Bowditch, *Relatively hyperbolic groups*, Int. J. Algebra Comput. (2012), no. 22, p1250016.
6. E. Chesebro, J. DeBlois, and H. Wilton, *Some virtually special hyperbolic 3-manifold groups*, Commentarii Mathematici Helvetici **87** (2012), no. 3, 727–787.
7. D. Cooper, D. Long, and A. Reid, *Essential surfaces in bounded 3-manifolds*, J. Amer. Math. Soc. **10** (1997), 553–564.
8. D. Cooper and D.D. Long, *Some surface subgroups survive surgery*, Geom. Topol. **5** (2001), 347–367.
9. F. Dahmani, *Combination of convergence groups*, Geom. Topol. **7** (2003), 933–963.
10. C. Drutu and M. Sapir, *Tree-graded spaces and asymptotic cones of groups*, Topology **44** (2005), no. 5, 959–1058, With an appendix by D. Osin and M. Sapir.
11. B. Farb, *Relatively hyperbolic groups*, Geom. Funct. Anal. **8** (1998), no. 5, 810–840.
12. B. Freedman and M. Freedman, *Kneser-haken finiteness for bounded 3-manifolds, locally free groups, and cyclic covers*, Topology **37** (1998), no. 1, 133–147.
13. F. Gauthier, *Geodesics in trees of hyperbolic and relatively hyperbolic spaces*, arXiv:0710.4079, 2007.
14. V. Gerasimov, *Floyd maps for relatively hyperbolic groups*, Geom. Funct. Anal. (2012), no. 22, 1361 – 1399.
15. V. Gerasimov and L. Potyagailo, *Quasiconvexity in the relatively hyperbolic groups*, arXiv:1103.1211, 2011.
16. E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques d’après mikhael gromov*, Progress in Math., Birkhäuser, 1990.
17. R. Gitik, *On the profinite topology on negatively curved groups*, J. Algebra (1999), no. 1, 80–86.
18. ———, *Ping-pong on negatively curved groups*, J. Algebra (1999), no. 1, 65–72.
19. M. Gromov, *Hyperbolic groups*, Essays in group theory (S Gersten, editor), vol. 1, pp. 75–263, Springer New York, 1987.
20. E. Hamilton, H. Wilton, and P. Zalesskii, *Separability of double cosets and conjugacy classes in 3-manifold groups*, arXiv:1109.2765, to appear in J. London Math. Soc., 2011.
21. G. Hruska, *Relative hyperbolicity and relative quasiconvexity for countable groups*, Algebr. Geom. Topol. **10** (2010), 1807–1856.
22. J. Manning and E. Martinez-Pedroza, *Separation of relatively quasiconvex subgroups*, Pacific Journal of Mathematics **244** (2010), no. 2, 309–334.
23. E. Martinez-Pedroza, *Combination of quasiconvex subgroups of relatively hyperbolic groups*, Groups, Geometry, and Dynamics **3** (2009), 317–342.
24. E. Martinez-Pedroza and A. Sisto, *Virtual amalgamation of relatively quasiconvex subgroups*, arXiv:1203.5839, 2012.

25. A. Minasyan, *Separable subsets of gferf negatively curved groups*, J. Algebra (2006), no. 2, 1090–1100.
26. M. Mj and L. Reeves, *A combination theorem for strong relative hyperbolicity*, Geom. Topol. (2008), 1777–1798.
27. D. Osin, *Elementary subgroups of relatively hyperbolic groups and bounded generation*, Internat. J. Algebra Comput. **16** (2006), no. 1, 99–118.
28. ———, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, vol. 179, Mem. Amer. Math. Soc., 2006.
29. D. Wise, *The structure of groups with quasiconvex hierachy*, available at <http://www.math.mcgill.ca/wise/papers.html>, 2011.

LE DPARTEMENT DE MATHMATIQUES DE LA FACULT DES SCIENCES D’ORSAY, UNIVERSIT PARIS-SUD 11, FRANCE

E-mail address: yabziz@gmail.com